Section 14.1: Multivariable functions 9/27/21

Definition: A multivariable function (of n variables with real values) is a function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$

function's name   Domain   number of variables

$\text{dom}(f)$ = domain of f

$\text{ran}(f) = \{ f(\vec{x}) : \vec{x} \in \text{dom}(f) \}$

NB: Often we won't explicitly state the domain of a function given formulaically. We'll use "the natural domain" in that case, i.e. the set of all inputs w/ defined outputs given by the formula

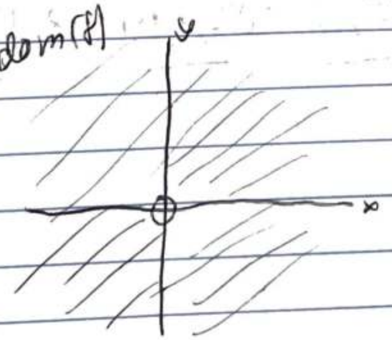
Ex: $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$

$\text{dom}(f) = \{ (x, y) \in \mathbb{R}^2 : \frac{x^2 - y^2}{x^2 + y^2} \text{ is defined} \}$

$= \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 0 \}$

$= \{ (x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0) \}$

dom(f)



Ex: $f(x,y) = \dfrac{x^2+y^2}{x^2-y^2}$

$$\mathrm{dom}(f) = \left\{(x,y) : \frac{x^2+y^2}{x^2-y^2} \text{ is defined}\right\}$$

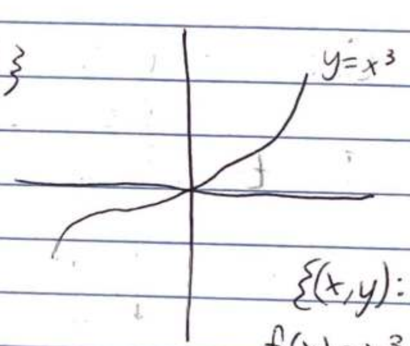
$$= \{(x,y) \in \mathbb{R}^2 : x^2-y^2 \neq 0\}$$

$$= \{(x,y) \in \mathbb{R}^2 : x \neq \pm y\}$$

$$= \{(x,y) \in \mathbb{R}^2 : |x| \neq |y|\}$$

Definition: The <u>graph</u> of a function f is

$$\mathrm{graph}(f) = \{(\vec{x}, f(\vec{x})) : \vec{x} \in \mathrm{dom}(f)\}$$



$\{(x,y) : y = x^3\}$

$f(x) = x^3$ //

$\{(x, f(x)) : x \in \mathrm{dom}(f)\}$

Ex: What is the shape of $f(x,y) = \sqrt{x^2+y^2+1}$?

Solution: Setting $z = f(x,y)$

$z = \sqrt{x^2+y^2+1}$ ie $z^2 = x^2+y^2+1$ and $z \geq 0$

ie $-x^2-y^2+z^2 = 1$ and $z \geq 0$

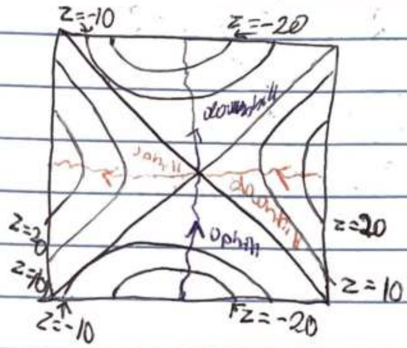
two-sheet hyperboloid

$\therefore$ graph(f) is the upper sheet of a two-sheet hyperboloid

Q: How can we represent ~~multivariable~~ 2-variable functions in 2-space?

A: Build a contour map (ie. level curves or elevation map)

Ex:



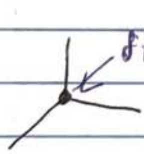
← contour map of a hyperbolic paraboloid

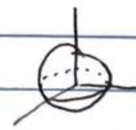
Ex: The unit hyper sphere is

$$S^3 = \{(x,y,z,t) \in \mathbb{R}^4 : x^2+y^2+z^2+t^2 = 1\}$$

$|t| \leq 1$

The t-level sets look like:

$t=-1$:  starts w/ a point

$t=0$: 

sphere gets bigger

$t=-\frac{1}{2}$: 

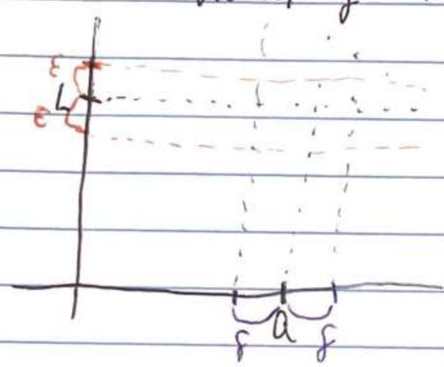
$t=\frac{1}{2}$:  gets smaller

$t=1$: 

Section 14.2: Limits and Continuity

In Calc III, the formal definition of a limit is:

Definition: Let f be a multivariable function and let $\vec{a} \in \mathbb{R}^n$ be a limit point of dom(f). The limit as $\vec{x}$ tends to $\vec{a}$ of f is $L \in \mathbb{R}$ when for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $\vec{a} \neq \vec{x} \in \text{dom}(f)$ we have $|\vec{x} - \vec{a}| < \delta$ implies $|f(\vec{x}) - L| < \varepsilon$



Notation

$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$

or

$f(\vec{x}) \to L$ as $\vec{x} \to \vec{a}$

Calc III version of "one-sided limits are equal"

Prop (Curves Criterion): Let f be a function and $\vec{a}$ a limit point of its domain $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$

if and only if for all curves $\vec{r}(t)$ in dom(f) s.t. $\lim_{t \to 0^+} \vec{r}(t) = \vec{a}$ we have $\lim_{t \to 0^+} f(\vec{r}(t)) = L$

Ex: Show $\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist

Solution: Let $f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$ and $l_{a,b}(t) = \langle at, bt \rangle$

Note that $\lim_{t \to 0} l_{a,b}(t) = \langle 0, 0 \rangle$

for all $t \neq 0$ we have

$f(l_{a,b}(t)) = \frac{(at)^2 - (bt)^2}{(at)^2 + (bt)^2} = \frac{(a^2 - b^2)t^2}{(a^2 + b^2)t^2} = \frac{a^2 - b^2}{a^2 + b^2}$

$\therefore \lim_{t \to 0} f(l_{0,1}(t)) = \lim_{t \to 0} \frac{a^2 - b^2}{a^2 + b^2}\Big|_{\substack{a=0 \\ b=1}} = \frac{0 - 1}{0 + 1} = -1$, $\lim_{t \to 0} f(l_{1,1}(t)) = 0 \neq -1$

$\therefore$ by curves criterion it does not exist